A coordinate-free treatment of the minimax and maximini theorems for eigenvalues of self-adjoint operators on a Hilbert space

1975 J. Phys. A: Math. Gen. 8 1853

(http://iopscience.iop.org/0305-4470/8/12/002)

J. Phys. A: Math. Gen., Vol. 8, No. 12, 1975. Printed in Great Britain. © 1975

A coordinate-free treatment of the minimax and maxmini theorems for eigenvalues of self-adjoint operators on a Hilbert space

C S Sharma and S SriRankanathan
Department of Mathematics, Birkbeck College, Malet Street, London WC1E 7HX, UK

Received 2 June 1975, in final form 29 July 1975

Abstract. The power and elegance of coordinate-free analysis is demonstrated by establishing the minimax and maxmini theorems for eigenvalues of self-adjoint operators on a Hilbert space as trivial corollaries of a series of lemmas of astounding simplicity. Furthermore the analysis brings out the essential identity of the two results in contradistinction to what has been generally believed before. A host of other related results are also established and in particular we have new simplified proofs of the inequalities of Weyl, Poincaré and Bateman; the latter two of these are better known to physicists as the Hylleraas–Undheim and MacDonald theorems and are extensively used in establishing upper bound and convergence properties of energy eigenvalues of stationary states of quantum systems.

1. Introduction

In a recent publication Stenger (1970) claimed that the minimum–maximum principle (Pólya 1954) and the maximum–minimum principle (Courant and Hilbert 1953) for eigenvalues of self-adjoint operators on a Hilbert space are two fundamentally different principles and are not related by a trivial interchange effected by re-ordering of eigenvalues. Stenger's assertion is obviously untrue in finite dimension. We now use the method of coordinate-free analysis to obtain new and elegant proofs of the two principles; the elegance derives from the extreme simplicity of the proofs. In our work basically the same lemma yields both principles thus showing their essential identity.

The basis of the minimum–maximum principle is the Poincaré (1890) inequality and this inequality is extensively used in establishing upper bound properties of variational estimates of energy eigenvalues of low-lying excited states of stationary quantum systems. In fact this inequality was rediscovered for the physicists by Hylleraas and Undheim (1930). Bateman (1912) had obtained a more powerful generalization of Poincaré's inequality and MacDonald (1933) presented Bateman's result in a form more easily accessible to quantum physicists, and this result is now commonly known as MacDonald's theorem. Bateman's result itself was in fact a generalization of an old result known to several nineteenth century mathematicians (cf MacDonald 1933). The maximum–minimum principle is based on Weyl's (1911) inequality and though this principle is often quoted in physics texts it has so far found little practical application. Most of the proofs of these principles and inequalities use both coordinates and determinants and are quite clumsy. In Stenger's (1970) simplified proofs it has been possible to dispense with the use of determinants though coordinates are still used and a knowledge of the theory of equations is assumed which in this context presumably depends on the theory of determinants.

We now present what seems to us the first coordinate-free proofs of these results. Our proofs turn out to be trivial and indeed as knowledge progresses there is so much more to learn that trivialization of clumsy proofs of profound results should be of very welcome assistance in one's journey to the frontiers of knowledge. This indeed is our main achievement in this work.

The coordinate-free analysis developed here is a continuation of the work of Sharma and co-workers (Sharma and Rebelo 1973a, b. 1975, Mare 1975a, b, c).

2. Formalities

Let $\mathscr{H}$ be a Hilbert space over the real or complex field. Let A be a self-adjoint operator on $\mathscr{H}$: it is not assumed that A is bounded. We shall denote the domain of A by $\mathscr{D}_A$ and the spectrum of A by $\operatorname{Sp} A$. We shall say that the spectrum of A is of type $\mathbf{H}$ if $\operatorname{Sp} A$ is bounded below, the lower part of the spectrum is purely discrete, and the first $\mathbf{N}$ points of the spectrum ordered to form an increasing enumeration have finite multiplicities (here $\mathbf{N}$ is either a positive integer or the cardinality $\aleph_0$ of the set of positive integers). We shall denote the eigenvalues in this enumeration by λ_i^A and the corresponding multiplicity by m_i^A. Let E_i^A be the orthogonal projection on the eigenspace $\mathscr{E}_i^A$ belonging to the eigenvalue λ_i^A, noting that the dimension of $\mathscr{E}_i^A$ is m_i^A. We denote the orthogonal complement of $\oplus_{i=1}^{\mathbf{N}} \mathscr{E}_i^A$ by $\mathscr{E}_c^A$ and the orthogonal projection on $\mathscr{E}_c^A$ by E_c^A. We have thus the following decompositions of $\mathscr{H}$ and the identity operator I on $\mathscr{H}$:

$$\mathscr{H} = \left(\bigoplus_{i=1}^{\mathbf{N}} \mathscr{E}_i^A\right) \oplus \mathscr{E}_c^A$$

$$I = \left(\sum_{i=1}^{\mathbf{N}} E_i^A\right) + E_c^A.$$

For $n \leqslant \mathbf{N}$ we define

$$F_n^A = \sum_{i=1}^{n} E_i^A$$

and the image of F_n^A is a subspace which we denote by $\mathscr{F}_n^A$. We denote the dimension of $\mathscr{F}_n^A$ by d_n^A: clearly

$$d_n^A = \sum_{i=1}^{n} m_i^A.$$

In our treatment eigenvalues are arranged to form an increasing enumeration and each eigenvalue is counted just once. However, people using coordinate-dependent treatments find it convenient to use an enumeration in which an eigenvalue λ_i of multiplicity m_i is counted m_i times. We shall call this alternative enumeration the *primed enumeration* and eigenvalues in this enumeration will carry a prime each. For a given $j \in \mathbb{Z}^+$ the two enumerations are related by

$$\lambda_j' = \lambda_i$$

where i is the smallest integer for which $j \leqslant d_i$. For a given i there are m_i different values of j for which the above relation holds and these values of j are $d_{i-1}+1$, $d_{i-1}+2, \ldots,$

$d_i - 1$, d_i. The primed enumeration from the point of view of coordinate-free analysis is wasteful and clumsy; nevertheless we use this to establish contact with the classical formulation.

Let P be the orthogonal projection on a subspace $\mathscr{P}$ of $\mathscr{H}$. We denote the orthogonal complement of $\mathscr{P}$ by $\mathscr{P}^\perp$ and the projection on $\mathscr{P}^\perp$ by $P^\perp$. Thus

$$P^\perp = I - P$$

and

$$\mathscr{P}^\perp = P^\perp(\mathscr{H}).$$

A self-adjoint operator A with spectrum of type **H** if it is unbounded is not defined on the whole of $\mathscr{H}$, but its domain $\mathscr{D}_A$ of definition forms a dense manifold in $\mathscr{H}$. We use the notation $I_{\mathscr{D}_A}$ to denote the restriction of the identity operator to $\mathscr{D}_A$. We shall further assume that $\lambda_{\mathbf{N}}^A < 0$; this is not a restriction of any consequence because in the general case we can always define an operator A' by

$$A' = A - \lambda I_{\mathscr{D}_A}$$

where $\lambda > \lambda_{\mathbf{N}}^A$. The spectrum of A' is obtained by translating each point in the spectrum of A by $-\lambda$ and the properties we wish to establish in a particular enumeration are shared by both A and A'.

It is usual to define a partial ordering of symmetric operators on $\mathscr{H}$ in the following way: we define

$$A_1 \leqslant A_2$$

if and only if $\mathscr{D}_{A_1} \subset \mathscr{D}_{A_2}$ *and*

$$\langle u, A_1 u\rangle \leqslant \langle u, A_2 u\rangle \qquad \forall u \in \mathscr{D}_{A_1}$$

where $\langle \cdot, \cdot \rangle$ is the positive definite Hermitian form on $\mathscr{H}$.

Let $\mathscr{P}$ be an n-dimensional subspace of $\mathscr{H}$ and let $\mathscr{P} \subset \mathscr{D}_A$. Let P be the orthogonal projection on $\mathscr{P}$. PAP is a well defined self-adjoint operator with domain $\mathscr{H}$; the spectrum of PAP consists of 0, which is an eigenvalue of infinite multiplicity to which all vectors in $\mathscr{P}^\perp$ belong, and r eigenvalues

$$\sigma_1^{PAP} < \ldots < \sigma_i^{PAP} < \ldots < \sigma_r^{PAP}$$

where the ith eigenvalue has multiplicity m_i^{PAP} such that

$$\sum_{i=1}^{r} m_i^{PAP} = n$$

and the direct sum of the eigenspaces belonging to these eigenvalues is $\mathscr{P}$. Note that if the kernel of PAP (denoted by Ker PAP) has a nonzero intersection with $\mathscr{P}$, then one of the σ_i^{PAP} is also zero and its multiplicity m_i^{PAP} is taken to be the dimension of (Ker PAP) $\cap\, \mathscr{P}$. More precisely σ_i^{PAP} are the eigenvalues of the restriction of PAP to $\mathscr{P}$.

3. Some lemmas

In what follows A is a self-adjoint operator on a Hilbert space over $\mathbb{C}$ or $\mathbb{R}$ and Sp A is of type H.

Lemma 3.1.

(i) $\lambda_1^A I_{\mathscr{D}_A} \leqslant A$

(ii) $\lambda_{i+1}^A I_{\mathscr{D}_A} \leqslant F_i^{A\perp} A F_i^{A\perp} \qquad (i < \mathbf{N}).$

Proof. Both are trivial consequences of the spectral theorem. To get the second inequality one has merely to observe that the spectral decomposition of $F_i^\perp A F_i^\perp$ is obtained by simply replacing the first i eigenvalues in the spectrum of A by 0 and recalling the assumption that all such eigenvalues are less than 0.

Lemma 3.2. Rayleigh–Ritz variation principle (Rayleigh 1945, Ritz 1909)

(i) $\lambda_1^A = \min_{\|u\|=1} \langle u, Au \rangle \qquad u \in \mathscr{D}_A$

(ii) $\lambda_{i+1}^A = \min_{\|u\|=1} \langle u, F_i^{A\perp} A F_i^{A\perp} u \rangle \qquad u \in \mathscr{D}_A.$

Proof. Trivial consequences of lemma 3.1.

Lemma 3.3. Let $\mathscr{V}$ be any vector space (finite- or infinite-dimensional) over any field. Let $\mathscr{U}$ be an n-dimensional subspace of $\mathscr{V}$ and let $\mathscr{W}$ be a subspace of $\mathscr{V}$ of dimension greater than n. Let $L \in \mathrm{Hom}(\mathscr{V}, \mathscr{V})$ be such that $L(\mathscr{V}) = \mathscr{U}$. Then there exists a nonzero vector $u \in \mathrm{Ker}\, L \cap \mathscr{W}$.

Proof. Suppose that the lemma is false; then for any collection u_i $(i = 1, \ldots, n+1)$ of $n+1$ linearly independent vectors in $\mathscr{W}$,

$$\sum_{i=1}^{n+1} \alpha_i (L u_i) = L\left(\sum_{i=1}^{n+1} \alpha_i u_i\right) = 0 \Rightarrow \alpha_i = 0 \qquad \text{for } i = 1, \ldots, n+1.$$

But $Lu_1, \ldots, Lu_{n+1}$ are $n+1$ vectors in an n-dimensional subspace $\mathscr{U}$ and cannot be linearly independent. The contradiction proves our assertion.

Lemma 3.4. Let $\mathscr{H}$ be any Hilbert space over any field. Let $\mathscr{P}$ be an n-dimensional subspace of $\mathscr{H}$ and let $\mathscr{G}$ be a subspace of $\mathscr{H}$ of dimension greater than n. Then there exists a nonzero vector $u \in \mathscr{G} \cap \mathscr{P}^\perp$.

Proof. Let P be the orthogonal projection on $\mathscr{P}$. Since $\mathrm{Ker}\, P = \mathscr{P}^\perp$ (Halmos 1957), in view of the preceding lemma there is nothing left to prove.

Lemma 3.5. Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and let SpA be of type **H**. Let $\mathscr{P}$ be an n-dimensional subspace of $\mathscr{H}$ such that $\mathscr{P} \subset \mathscr{D}_A$. Let r be the smallest integer such that $n < d_r^A$. Let P be the orthogonal projection on $\mathscr{P}$. Then the spectrum of $P^\perp A P^\perp$ lying below λ_{r+1}^A is non-empty, purely discrete and consists entirely of a finite number of point eigenvalues of finite multiplicities. Furthermore, the dimension of the direct sum of eigenspaces of $P^\perp A P^\perp$ belonging to eigenvalues in the interval $]-\infty, \lambda_r^A]$ is at least $d_r^A - n$.

Proof. Let E be the image of the interval $]-\infty, \lambda_{r+1}^A[$ under the spectral measure induced by $P^\perp A P^\perp$. First we prove that $E(\mathscr{H})$ is finite-dimensional. Suppose that $E(\mathscr{H})$ is infinite-dimensional. Let $x \in E(\mathscr{H})$; then

$$\langle x, P^{\perp}AP^{\perp}x\rangle = \langle x, EAEx\rangle < \langle x, \lambda_{r+1}^{A} I_{\mathscr{D}_A} x\rangle.$$

Since $\mathscr{F}_{r+1}^{A}$ is finite-dimensional, $E(\mathscr{H}) \cap \mathscr{F}_{r+1}^{A\perp}$ has a nonzero vector, say u, but

$$\langle u, Au\rangle > \langle u, \lambda_{r+1}^{A} I_{\mathscr{D}_A} u\rangle,$$

a contradiction which shows that $E(\mathscr{H})$ is indeed finite-dimensional. Since $E(\mathscr{H})$ is finite-dimensional, $P^{\perp}AP^{\perp}$ restricted to $E(\mathscr{H})$ is a self-adjoint operator on a finite-dimensional Hilbert space and therefore has a purely point spectrum which consists of a finite number of point eigenvalues of finite multiplicities. The spectrum of $P^{\perp}AP^{\perp}$ below λ_{r+1}^{A} is precisely the spectrum of $P^{\perp}AP^{\perp}$ restricted to $E(\mathscr{H})$. We still have to prove that the spectrum of $P^{\perp}AP^{\perp}$ below λ_{r+1}^{A} is non-empty, ie that E is nonzero. Since $n < d_r^A$ from lemma 3.4 there is a nonzero vector $u \in \mathscr{P}^{\perp} \cap \mathscr{F}_r^A$, for this vector

$$\langle u, P^{\perp}AP^{\perp}u\rangle = \langle u, Au\rangle \leqslant \langle u, \lambda_r^A I_{\mathscr{D}_A} u\rangle,$$

which is impossible if the spectrum of $P^{\perp}AP^{\perp}$ below λ_{r+1}^{A} is empty.

Let $\mathscr{G}$ be the direct sum defined in the last sentence of the lemma. Suppose that the last assertion is false and the dimension of $\mathscr{G}$ is less than $d_r^A - n$. Then $\mathscr{U} = \mathscr{P} \oplus \mathscr{G}$ has dimension less than d_r^A; let U be the projection on $\mathscr{U}$. It follows from lemma 3.4 that there exists a nonzero vector $u \in \mathscr{U}^{\perp} \cap \mathscr{F}_r^A$, for this vector

$$\langle u, Au\rangle = \langle u, U^{\perp}AU^{\perp}u\rangle \leqslant \langle u, \lambda_r^A I_{\mathscr{D}_A} u\rangle$$

which is possible only if $U^{\perp}AU^{\perp}$ has an eigenvalue less than or equal to λ_r^A, but since $U^{\perp}(\mathscr{H})$ is invariant under $P^{\perp}AP^{\perp}$, and furthermore since $U^{\perp} < P^{\perp}$, every eigenvalue of $U^{\perp}AU^{\perp}$ is also an eigenvalue of $P^{\perp}AP^{\perp}$. All eigenspaces of $P^{\perp}AP^{\perp}$ which belong to eigenvalues less than or equal to λ_r^A, however, are subspaces of $\mathscr{G}$ and therefore of $\mathscr{U}$. The contradiction proves the final assertion.

Lemma 3.6. The spectrum of $P^{\perp}AP^{\perp}$ of lemma 3.5 is bounded below and the infimum of the spectrum is a point eigenvalue which is the minimum value of $\{\langle u, Au\rangle : u \in \mathscr{P}^{\perp}\ \&\ \|u\| = 1\}$.

Proof. These are elementary consequences of lemmas 3.2 and 3.5.

4. The minimum–maximum theory

All properties in this section concern a self-adjoint operator A on a Hilbert space $\mathscr{H}$ over the complex (or real) field with SpA of type **H**.

Proposition 4.1. Let $\mathscr{P}$ be any n-dimensional subspace of $\mathscr{H}$ such that $\mathscr{P} \subset \mathscr{D}_A$ and let P be the orthogonal projection on $\mathscr{P}$. Let σ_i^{PAP} $(i = 1, \ldots, r)$ be an increasing enumeration of the eigenvalues of the restriction of PAP to $\mathscr{P}$. For a given $j \leqslant r$, let $d_j^{PAP} = \sum_{i=1}^{j} m_i^{PAP}$ and let s be the smallest integer for which

$$d_j^{PAP} \leqslant d_s^A.$$

Then $\lambda_s^A \leqslant \sigma_j^{PAP}$.

Proof. Let $\mathscr{F}_j^{PAP} = \oplus_{i=1}^{j} \mathscr{E}_i^{PAP}$ where $\mathscr{E}_i^{PAP}$ is the eigenspace of the restriction of PAP to

$\mathscr{P}$ belonging to the eigenvalue σ_i^{PAP}. Since s is the smallest integer for which $d_j^{PAP} \leqslant d_s^A$, $d_j^{PAP} > d_{s-1}^A$. Therefore (cf lemma 3.4) $\mathscr{F}_j^{PAP}$ contains at least one nonzero vector $u \in \mathscr{F}_{s-1}^{A\perp}$. But for any $u \in \mathscr{F}_j^{PAP}$

$$\langle u, PAPu\rangle = \langle u, Au\rangle \leqslant \sigma_j^{PAP}\langle u, I_{\mathscr{D}_A}u\rangle$$

and for any $u \in \mathscr{F}_{s-1}^{A\perp}$.

$$\langle u, Au\rangle \geqslant \lambda_s^A\langle u, I_{\mathscr{D}_A}u\rangle$$

which shows the impossibility of $\sigma_j^{PAP} < \lambda_s^A$.

Corollary 4.1.1. Poincaré's (1890) inequality. With the primed enumeration of eigenvalues in proposition 4.1, for $j \leqslant n$

$$\lambda_j'^A \leqslant \sigma_j'^{PAP}.$$

Proof. By the definition of the primed enumeration $\lambda_j'^A = \lambda_i^A$ where i is the smallest integer for which $j \leqslant d_i^A$ and $\sigma_j'^{PAP} = \sigma_s^{PAP}$ where s is the smallest integer for which $j \leqslant d_s^{PAP}$. The smallest integer t for which $d_s^{PAP} \leqslant d_t^A$ cannot be less than i, whence

$$\lambda_j'^A = \lambda_i^A \leqslant \lambda_t^A \leqslant \sigma_s^{PAP} = \sigma_j'^{PAP}.$$

In atomic physics this result is known as the Hylleraas–Undheim theorem.

Corollary 4.1.2. MacDonald's (1933) theorem. Let $\mathscr{Q} \subset \mathscr{D}_A$ be an $(n+1)$-dimensional subspace of $\mathscr{H}$ which contains $\mathscr{P}$. Let Q be the orthogonal projection on $\mathscr{Q}$. Let $\sigma_i'^{QAQ}$ $(i = 1, \ldots, n+1)$ be the primed enumeration of the eigenvalues of QAQ to $\mathscr{Q}$. Then

$$\sigma_1'^{QAQ} \leqslant \sigma_1'^{PAP} \leqslant \sigma_2'^{QAQ} \leqslant \ldots \leqslant \sigma_n'^{PAP} \leqslant \sigma_{n+1}'^{QAQ}.$$

Proof. Regarding $\mathscr{Q}$ as the whole Hilbert space and QAQ as A and applying Poincaré's inequality, we have

$$\sigma_i'^{QAQ} \leqslant \sigma_i'^{PAP} \qquad (i = 1, \ldots, n).$$

Replacing A by $-A$ and denoting the corresponding eigenvalues by $(-\sigma)_i'^{QAQ}$ and $(-\sigma)_i'^{PAP}$ respectively, we have

$$(-\sigma)_i'^{QAQ} = -\sigma_{n+2-i}'^{QAQ} \qquad (i = 1, \ldots, n+1)$$

and

$$(-\sigma)_i'^{PAP} = -\sigma_{n+1-i}'^{PAP} \qquad (i = 1, \ldots, n).$$

Using Poincaré's inequality again we get

$$-\sigma_{n+2-i}'^{QAQ} = (-\sigma)_i'^{QAQ} \leqslant (-\sigma)_i'^{PAP} = -\sigma_{n+1-i}'^{PAP}$$

which implies that

$$\sigma_{n+1-i}'^{PAP} \leqslant \sigma_{n+2-i}'^{QAQ} \qquad (i = 1, \ldots, n)$$

or

$$\sigma_i'^{PAP} \leqslant \sigma_{i+1}'^{QAQ} \qquad (i = 1, \ldots, n).$$

Combining the first and last inequalities we have MacDonald's theorem.

Proposition 4.2. The minimum–maximum theorem. For a given $r \in \mathbb{Z}^+$ let n be any integer in the interval $]d^A_{r-1}, d^A_r]$. Let $\mathbf{P}$ be the family of n-dimensional subspaces of $\mathscr{H}$ which are also subsets of $\mathscr{D}_A$. Then

$$\lambda^A_r = \min_{\mathscr{P} \in \mathbf{P}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}\}.$$

Proof. For a given $\mathscr{P} \in \mathbf{P}$, $\max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}\}$ is, of course, the largest eigenvalue σ^{PAP}_j of PAP restricted to $\mathscr{P}$, where P is the orthogonal projection on $\mathscr{P}$. Clearly $d^{PAP}_j = n$ and the smallest integer i for which $d^{PAP}_j \leqslant d^A_i$ is r, whence

$$\lambda^A_r \leqslant \sigma^{PAP}_j = \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}\}.$$

Next let $\mathscr{P}$ be $\mathscr{F}^A_{r-1} \oplus \mathscr{G}$ where $\mathscr{G}$ is any $(n - d^A_{r-1})$-dimensional subspace of $\mathscr{E}^A_r$. Clearly $\mathscr{P} \subset \mathscr{D}_A$ and is n-dimensional, so that $\mathscr{P} \in \mathbf{P}$. In this case

$$\langle u, Au\rangle \leqslant \langle u, \lambda^A_r I_{\mathscr{D}_A} u\rangle$$

so that $\lambda^A_r \geqslant \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}\}$. The combination of the two inequalities yields the desired result.

Corollary 4.2.1. For the primed enumeration of eigenvalues in proposition 4.2

$$\lambda'^A_n = \min_{\mathscr{P} \in \mathbf{P}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}\}.$$

Proof. This is an immediate consequence of the preceding proposition and the definition of the primed enumeration. (Note that for the purposes of this corollary n is any positive integer and r of the preceding proposition is defined with the help of n in such a way that the given relation between the two is preserved.)

5. The maximum–minimum theory

All properties in this section also concern a self-adjoint operator A on a Hilbert space $\mathscr{H}$ over the complex (or real) field with SpA of type $\mathbf{H}$.

Proposition 5.1. Let $\mathscr{P}$ be an n-dimensional subspace of $\mathscr{H}$ and a subset of $\mathscr{D}_A$ and let r be the smallest integer for which $n \leqslant d^A_r$. Then

$$\min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp\} \leqslant \lambda^A_{r+1}.$$

Proof. An immediate consequence of lemma 3.5.

Corollary 5.1.1. Weyl's (1911) inequality. For the primed enumeration of eigenvalues in proposition 5.1

$$\min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp\} \leqslant \lambda'^A_{n+1}.$$

Proof. It follows immediately from the preceding proposition and the definition of the primed enumeration.

Proposition 5.2. The maximum–minimum theorem. For a given positive integer r, let n

be any integer in the interval $[d^A_{r-1}, d^A_r - 1]$. Let $\mathbf{P}$ be the family of n-dimensional subspaces of $\mathscr{H}$ which are also subsets of $\mathscr{D}_A$. Then

$$\lambda^A_r = \max_{\mathscr{P}\in \mathbf{P}} \min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp\}.$$

Proof. Let $\mathscr{P} \in \mathbf{P}$ and let P be the orthogonal projection on $\mathscr{P}$. Since $d^A_r - n > 1$, it follows from lemma 3.5 that $P^\perp A P^\perp$ has at least one eigenvalue which is less than or equal to λ^A_r; we can, therefore, particularly in view of lemma 3.6, assert that

$$\min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp\} \leqslant \lambda^A_r.$$

On the other hand $\mathscr{F}^A_r$ is of dimension greater than n and the dimension of $\mathscr{F}^A_{r-1}$ is either less than or equal to n, hence there is an n-dimensional subspace $\mathscr{P}$ of $\mathscr{F}^A_r$ such that $\mathscr{F}^A_{r-1} \subset \mathscr{P}$. Clearly $\mathscr{P} \in \mathbf{P}$ and for this choice of $\mathscr{P}$, $\mathscr{P}^\perp \cap \mathscr{E}^A_r$ is non-trivial (cf lemma 3.4) and for any unit vector in this intersection

$$\langle u, Au\rangle = \lambda^A_r,$$

so that $\max \min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp, \mathscr{F}^A_{r-1} \subset \mathscr{P}, \mathscr{P} \in \mathbf{P}\} \geqslant \lambda^A_r$. The two inequalities together yield the desired result.

Corollary 5.2.1. Let $\mathbf{P}$ be the family of n-dimensional subspaces of $\mathscr{H}$ which are also subsets of $\mathscr{D}_A$. Then for the primed enumeration of eigenvalues of A,

$$\lambda'^A_{n+1} = \max_{\mathscr{P}\in \mathbf{P}} \min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{P}^\perp\}.$$

Proof. See the preceding proposition and the definition of the primed enumeration.

6. Some other results

The methods developed above make it easy to find simplified proofs of a variety of results on the extremal characterization of eigenvalues of self-adjoint operators with spectrum of type $\mathbf{H}$. To substantiate our claim we give here proofs of two theorems which are interesting.

Proposition 6.1. The maximum–minimum–maximum theorem. Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ over the complex (or real) field, SpA being of type $\mathbf{H}$. Let $\mathbf{I}$ and $\mathbf{J}$ respectively be families of i- and j-dimensional subspaces of $\mathscr{H}$ which are also subsets of $\mathscr{D}_A$. Let r be the smallest integer such that $i+j \leqslant d^A_r$. Then

$$\lambda^A_r = \max_{\mathscr{J}\in \mathbf{J}} \min_{\substack{\mathscr{I}\in \mathbf{I}\\ \mathscr{I}\subset \mathscr{J}^\perp}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{I}\}.$$

Proof. For a given $\mathscr{J} \in \mathbf{J}$, let J be the orthogonal projection on $\mathscr{J}$. Lemma 3.5 tells us that the direct sum $\mathscr{G}$ of eigenspaces of $J^\perp A J^\perp$ belonging to eigenvalues in the interval $]-\infty, \lambda^A_r]$ is at least $(d_r - j > i)$-dimensional. Hence it is possible to choose an $\mathscr{I} \in \mathbf{I}$, $\mathscr{I} \subset \mathscr{J}^\perp$ such that $\forall u \in \mathscr{I}$ (any i-dimensional subspace of $\mathscr{G}$ will do) $\langle u, Au\rangle < \lambda^A_r \|u\|^2$, hence

$$\min_{\substack{\mathscr{I}\in \mathbf{I}\\ \mathscr{I}\subset \mathscr{J}^\perp}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{I}\} \leqslant \lambda^A_r.$$

On the other hand if $\mathscr{J}$ is chosen to be any j-dimensional subspace of $\mathscr{F}_r^A$ and $\mathscr{I}$ is chosen to be any i-dimensional subspace of $\mathscr{J}^\perp \cap \mathscr{F}_r^A$, then since $\mathscr{I} \oplus \mathscr{J}$ is of dimension greater than d_{r-1}^A, $\mathscr{I}$ contains at least one unit vector in the eigenspace $\mathscr{E}_r^A$ and for this choice of $\mathscr{J}$

$$\min_{\substack{\mathscr{I} \subset \mathscr{F}_r^A \cap \mathscr{J}^\perp \\ \mathscr{I} \in \mathrm{I}}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{I}\} = \lambda_r^A$$

so that

$$\max_{\mathscr{J} \in \mathrm{J}} \min_{\substack{\mathscr{I} \in \mathrm{I} \\ \mathscr{I} \subset \mathscr{J}^\perp}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{I}\} \geqslant \lambda_r^A.$$

The two inequalities give us the required result.

Corollary 6.1.1. For the primed enumeration of eigenvalues in the preceding proposition (Stenger 1968)

$$\lambda_{i+j}'^A = \max_{\mathscr{J} \in \mathrm{J}} \min_{\substack{\mathscr{I} \in \mathrm{I} \\ \mathscr{I} \subset \mathscr{J}^\perp}} \max_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{I}\}.$$

Proof. See the preceding proposition and the definition of the primed enumeration.

Proposition 6.2. Let A and B be two self-adjoint operators on a Hilbert space $\mathscr{H}$ over the complex (or real) field such that (i) SpA and SpB are both of type $\mathbf{H}$, (ii) $\mathscr{D}_A \subset \mathscr{D}_B$ and (iii) $\langle u, Bu\rangle \leqslant \langle u, Au\rangle\, \forall u \in \mathscr{D}_A$. For any given positive integer k, let r and s be the smallest integers such that $k \leqslant d_r^B$ and $k \leqslant d_s^A$. Then $\lambda_r^B \leqslant \lambda_s^A$.

Proof. Suppose $\lambda_s^A < \lambda_r^B$. Let λ_j^B be the largest eigenvalue of B which is less than or equal to λ_s^A. Such an eigenvalue of B exists because the set of eigenvalues of B less than or equal to λ_s^A is not empty, as can be seen from the following:

$$\lambda_1^B = \min_{\|u\|=1} \{\langle u, Bu\rangle : u \in \mathscr{D}_B\} < \min_{\|u\|=1} \{\langle u, Au\rangle : u \in \mathscr{D}_A\} = \lambda_1^A \leqslant \lambda_s^A$$

and is bounded above by λ_r^B. Clearly $d_j^B < k \leqslant d_s^A$, hence there is a nonzero vector $u \in \mathscr{F}_j^{B\perp} \cap \mathscr{F}_s^A$, for this u,

$$\lambda_{j+1}^B \|u\|^2 \leqslant \langle u, Bu\rangle \leqslant \langle u, Au\rangle \leqslant \lambda_s^A \|u\|^2,$$

which is clearly absurd. The contradiction proves our assertion.

Corollary 6.2.1. For the primed enumeration of eigenvalues in the preceding proposition (Weyl 1950, Kato 1955, Stenger 1970)

$$\lambda_k'^B \leqslant \lambda_k'^A.$$

Proof. See the preceding proposition and the definition of the primed enumeration.

7. Concluding remarks

We wish to make a few more points. First, the minimum–maximum theory helps us to find upper bounds to energy and other eigenvalues of stationary pure states in quantum theory. For any n-dimensional subspace $\mathscr{P} \subset \mathscr{D}_A$, PAP is effectively a self-adjoint operator on a finite-dimensional Hilbert space and its n eigenvalues provide upper bounds to the first n eigenvalues of A in the primed enumeration. The maximum–minimum theory replaces the problem of finding eigenvalues of A by that of $P^{\perp}AP^{\perp}$ which, like A, is a self-adjoint operator on an infinite-dimensional Hilbert space and so for practical purposes is of little use. Second, that most of the results are essentially corollaries of lemma 3.3 which is a very powerful yet trivial result. Third, that the main use of the maximum–minimum theory is through the insight which it provides with the help of lemmas 3.5 and 3.6 into the spectrum of $P^{\perp}AP^{\perp}$. It could also be useful in the study of the Feshbach formalism for decaying states in quantum theory where the underlying Hilbert space is partitioned into orthogonal subspaces $\mathscr{P}$ and $\mathscr{P}^{\perp}$ (see, eg, Sharma and Bowtell 1973), though the situation in the case of decaying states is substantially more complicated because for any sensible choice of $\mathscr{P}$ both $\mathscr{P}$ and $\mathscr{P}^{\perp}$ are infinite-dimensional.

References

Bateman H 1912 *Bull. Am. Math. Soc.* **18** 179–82
Courant R and Hilbert D 1953 *Methods of Mathematical Physics* vol 1 (New York: Interscience)
Halmos P 1957 *Introduction to Hilbert Space and the Theory of Spectral Multiplicity* (New York: Chelsea)
Hylleraas E A and Undheim B 1930 *Z. Phys.* **65** 759–72
Kato T 1955 *Technical Report No* 7 (Berkeley: Department of Matematics, University of California)
MacDonald J K L 1933 *Phys. Rev.* **43** 830–3
Mare S B W 1975a *Phys. Lett.* **53A** 8
—— 1975b submitted for publication
—— 1975c submitted for publication
Poincaré H 1890 *Am. J. Math.* **12** 211–94
Pólya G 1954 *Studies in Mathematics and Mechanics* (New York: Academic Press) pp 200–7
Rayleigh Lord (Strutt J W) 1945 *The Theory of Sound* (New York: Dover)
Ritz W 1909 *J. Reine Agnew. Math.* **135** 1–61
Sharma C S and Bowtell G 1973 *Can. J. Phys.* **51** 1637–45
Sharma C S and Rebelo I 1973a *J. Phys.* A: *Math.. Nucl. Gen.* **6** 459–67
—— 1973b *Phys. Lett.* **44A** 29–30
—— 1975 *Int. J. Theor. Phys.* **36** No 6
Stenger W 1968 *J. Math. Mech.* **17** 641–8
—— 1970 *Inequalities* vol 2 (New York: Academic Press) pp 375–87
Weyl H 1911 *Math. Ann.* **71** 441–79
—— 1950 *Bull. Am. Math. Soc.* **56** 115–39